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Scalar and spinning particles in an external linearized gravitational wave field

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Abstract

We study the interaction of a scalar and a spinning particle with a coherent linearized gravitational wave field treated as a classical spin two external field. The spin degrees of freedom of the spinning particle are described by skew-commuting variables. We derive the explicit expressions for the eigenfunctions and the Green's functions of the theory. The discussion is exact within the approximation of neglecting radiative corrections and we prove that the result is completely determined by the semi-classical contribution: this is shown by comparing the wavefunctions with the (pseudo)classical solutions of the Hamilton–Jacobi equation.

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1. Introduction

For many years Grassmann variables have been a common practice for giving a *pseudoclassical* framework to describe spin [1–5], or other internal degrees of freedom of elementary particles [6]. Of course the value of such models must be thought of in view of their quantization, that can be the canonical one or, in view of the inherent pseudoclassical description, the path integral quantization. Although some technical caution had to be taken in extending the path integral to skew-commuting variables (it is, for instance, relevant whether their number is even or odd), nevertheless the programme was successfully carried out, see e.g., [7].

We have recently presented the quantization of the theory describing a spin $\frac{1}{2}$ particle in an external electromagnetic wave field by using the path integral and external source method and we have produced the corresponding Feynman propagator [8]. The electromagnetic interaction at the pseudoclassical level was introduced by the standard minimal coupling directly in the constraints describing the theory. The self-consistency of the theory automatically produced the non-minimal electromagnetic coupling characteristic for the spin $\frac{1}{2}$ particle (Pauli term). The calculations leading to the final result were straightforward, although somewhat complex, due to the above-mentioned cautions. In this paper, we consider the problem of describing the interaction of a scalar and a spinning particle at a first quantized level with an external gravitational field. Of course, if we do not want to go far beyond our possibilities of performing explicit calculations, the gravitational field must be taken in the linear approximation. Even with this simplifying assumption, however, we have to face new features with respect to [8], such as the treatment of covariant derivatives of Grassmann variables, responsible for the spin–gravitational field interaction and whose affine connections have to be substituted with their symmetric part, implying that the spinning particle cannot be directly coupled to the torsion [9]. Following the approach started by [10], we are able to evaluate the eigenfunctions and the Feynman kernel if we assume that the linearized field is that of a gravitational wave. When restricting ourselves to the even sector, we recover the propagator of a scalar particle, calculated in [11], by using symmetry arguments and canonical transformations in the path-integral framework.

Results in this direction have been known since some years. In [12], the solution for a Klein–Gordon particle interacting with a general plane gravitational wave was studied and the semi-classical nature of the wavefunction was observed. In [13], the solution for a Dirac particle in the background of a gravitational plane wave was derived by using coordinates similar to the light-cone ones. The scattering cross section was also calculated and it was found to be the same as in the scalar case, indirectly proving its semi-classical nature in this case too. A classical approach to the scattering of scalar and spinning test particles by gravitational plane waves was given in [14], by using the Mathisson–Papapetrou–Dixon equation. In the present paper, we want to revisite the analogies between the classical and the quantum problems, by unifying them in the framework of the pseudoclassical mechanics. Indeed, the use of the Hamilton–Jacobi theory with Grassmann variables will prove to be a very powerful tool for calculating the quantum mechanical wavefunctions in terms of the pseudoclassical action.

A brief summary of the paper is as follows. In section 2, we give the main definitions of the physical quantities as well as the gauge conditions we will assume on the metric and on the affine connections. We then calculate the wavefunction for the scalar particle, whose Green's function is discussed in section 3 together with its semi-classical limit. The linearized theory for a spinning particle in the external gravitational wave is described in section 4, while in the final section 5 we consider the corresponding Green's function and we show, for the spinning particles too, the semi-classical nature of the results.

2. Scalar particle in an external gravitational wave: linear theory

According to general relativity, the interaction of matter with a given gravitational field is introduced in a geometrical way by replacing the flat Minkowski metric $\eta_{\mu\nu} = (+, -, -, -)$ by a tensor $g_{\mu\nu}(x)$ depending upon the coordinates $x^\mu = (x^0, \vec{x}) \equiv (t, \vec{x})$ and by using the general covariance principle [15]. Therefore, the action describing a massive scalar particle interacting with an external gravitational field can be written as

$$S = \int \mathrm{d}^4x\, \sqrt{-g(x)}\mathcal{L}(\phi(x), D_\mu\phi(x)) \tag{2.1}$$

where $g(x)$ is the determinant of the metric and D_μ the covariant derivative. Letting $g^{\mu\nu}(x)$ be the inverse of the metric tensor, the Lagrangian density $\mathcal{L}$ has the form

$$\mathcal{L} = \tfrac{1}{2}g^{\mu\nu}(x)D_\mu\phi(x)D_\nu\phi(x) - \tfrac{1}{2}m^2\phi^2(x) \tag{2.2}$$

and produces the equation of motion

$$(g^{\mu\nu}(x)D_\mu D_\nu + m^2)\phi(x) = 0. \tag{2.3}$$

The energy–momentum tensor of the matter field, in the absence of gravity, is obtained by the usual relation

$$T_0^{\mu\nu}(x) = -\frac{2}{\sqrt{-g(x)}}\frac{\delta S}{\delta g_{\mu\nu}(x)}\bigg|_{g_{\mu\nu}(x)=\eta_{\mu\nu}} = -\eta^{\mu\nu}\mathcal{L}_0 + \partial^\mu\phi(x)\partial^\nu\phi(x) \tag{2.4}$$

where $\mathcal{L}_0$ is the free Lagrangian density. When the gravitational field is weak and described by a small perturbation to the flat metric

$$g_{\mu\nu}(x) \simeq \eta_{\mu\nu} + h_{\mu\nu}(x), \qquad g^{\mu\nu}(x) \simeq \eta^{\mu\nu} - h^{\mu\nu}(x) \tag{2.5}$$

we can consider the linearized theory with a Lagrangian density

$$\mathcal{L} = \mathcal{L}_0 - \tfrac{1}{2}h_{\mu\nu}(x)T_0^{\mu\nu}(x) = \left(1 + \tfrac{1}{2}h(x)\right)\mathcal{L}_0 - \tfrac{1}{2}h^{\mu\nu}(x)\partial_\mu\phi(x)\partial_\nu\phi(x) \tag{2.6}$$

where $h(x) = \eta_{\mu\nu}h^{\mu\nu}(x) = h^\mu_\mu(x)$. The Lagrange equations derived from (2.6) are thus

$$\left(\left(1 + \tfrac{1}{2}h(x)\right)(\partial^\mu\partial_\mu + m^2) - h^{\mu\nu}(x)\partial_\mu\partial_\nu - \left(\partial_\mu h^\mu_\nu(x) - \tfrac{1}{2}\partial_\nu h(x)\right)\partial^\nu\right)\phi = 0. \tag{2.7}$$

Upon multiplying on the left by the factor $(1 - h(x)/2)$, at this order in $h_{\mu\nu}(x)$ equation (2.7) can be further simplified to

$$\left(\partial^\mu\partial_\mu - h^{\mu\nu}(x)\partial_\mu\partial_\nu - \left(\partial_\mu h^\mu_\nu(x) - \tfrac{1}{2}\partial_\nu h(x)\right)\partial^\nu + m^2\right)\phi = 0. \tag{2.8}$$

From (2.8) it appears that we are considering a theory describing the interaction of scalar matter with a given gravitational field treated as a spin two external field in a flat Minkowski space–time [16].

As is well known that there are some conditions that can be imposed on the metric tensor, in analogy to the gauge choice for the vector potential $A_\mu(x)$ in electrodynamics. A particularly convenient requirement is represented by the harmonic condition [15]

$$g^{\mu\nu}(x)\Gamma^\lambda_{\mu\nu}(x) = 0 \tag{2.9}$$

whose linearization reads

$$\partial_\mu h^\mu_\nu(x) = \tfrac{1}{2}\partial_\nu h(x). \tag{2.10}$$

To our purpose it turns out to be even more convenient to choose a restriction of the general harmonic gauge, that is

$$\partial_\mu h^\mu_\nu(x) = 0, \qquad h(x) = 0. \tag{2.11}$$

This choice, in addition to some algebraic simplifications, also avoids the more serious ordering problem that would be present in some terms of equation (2.7). Finally, we consider the linearized gravitational field as the one produced by a wave of arbitrary spectral composition and polarization properties, but propagating in a fixed direction:

$$g_{\mu\nu}(x) \simeq \eta_{\mu\nu} + h_{\mu\nu}(\kappa_x), \qquad h_{\mu\nu}(\kappa_x) = a_{\mu\nu}f(\kappa_x), \qquad \kappa_x = k\cdot x. \tag{2.12}$$

Here, $k^2 = 0$ and $a_{\mu\nu} = a_{\nu\mu}$ is the polarization tensor. Due to these choices equation (2.11) then becomes

$$k_\mu a^\mu_\nu = 0, \qquad a^\mu_\mu = 0 \tag{2.13}$$

and it is easy to cast the linearized Klein–Gordon wave equation (2.7) into the form

$$(\partial^\mu\partial_\mu - h^{\mu\nu}(x)\partial_\mu\partial_\nu + m^2)\phi(x) = 0. \tag{2.14}$$

Remembering the derivation of the Volkov solution for the problem of spin $\frac{1}{2}$ particle in a electromagnetic wave field [8, 10], we look for a solution of (2.14) in the form

$$\phi_p(x) = (2p_0V)^{-\frac{1}{2}}\exp(-\mathrm{i}p\cdot x - \mathrm{i}F(\kappa_x)), \tag{2.15}$$

where p^μ is a constant 4-vector and $F(\kappa_x) \to 0$ as $x^0 \to -\infty$. Moreover, since $F(\kappa_x)$ is still an undetermined function, by adding to p^μ an arbitrary 4-vector proportional to k^μ we obtain a wavefunction $\phi_p(x)$ of the same form. Therefore, without loss of generality we can impose on the 4-vector p^μ the mass-shell condition $p^2 = m^2$. After substituting the solution (2.15) into equation (2.14) and taking into account the mass-shell relation, the null character of k^μ and the harmonic gauge condition (2.11) we obtain for $F(\kappa_x)$ the following first-order differential equation:

$$2k \cdot pF'(\kappa_x) - p_\mu p_\nu h^{\mu\nu}(\kappa_x) = 0 \tag{2.16}$$

where $F'(\kappa_x)$ is the derivative with respect to the argument. By integrating this first-order equation, from (2.16) we finally obtain for $\phi_p(x)$ the Volkov-like solution

$$\phi_p(x) = (2p_0V)^{-\frac{1}{2}} \exp\left(-\mathrm{i}p \cdot x - \frac{\mathrm{i}}{2k \cdot p}\int_{-\infty}^{\kappa_x} \mathrm{d}\kappa\, h^{\mu\nu}(\kappa) p_\mu p_\nu\right). \tag{2.17}$$

3. Green's function for the scalar particle

The standard way of obtaining the Green's function for the wave equation is to solve the non-homogeneous boundary value problem with an added delta-function source term (see, e.g., [17]):

$$(\partial^\mu\partial_\mu - h^{\mu\nu}(x)\partial_\mu\partial_\nu + m^2)\Delta_F(x, y) = -\delta^4(x - y). \tag{3.1}$$

By extracting the singularity due to the mass-shell condition, we can assume for the Green function the form

$$\Delta_F(x, y) = \int \frac{\mathrm{d}^4 p}{(2\pi)^4} \frac{\exp(-\mathrm{i}p \cdot (x - y))}{p^2 - m^2 + \mathrm{i}\epsilon} \exp(-\mathrm{i}F_1(\kappa_x, \kappa_y)) \tag{3.2}$$

where $\kappa_x = k \cdot x, \kappa_y = k \cdot y$. The complete Feynman's propagator $\Delta_F(x, y)$ is therefore a superposition of free propagators modulated by the phase factor $\exp\{-\mathrm{i}F_1(\kappa_x, \kappa_y)\}$. The equation for $F_1(\kappa_x, \kappa_y)$ is determined by (3.1), (3.2) and reads

$$\int \frac{\mathrm{d}^4 p}{(2\pi)^4} \frac{\exp(-\mathrm{i}p \cdot (x - y)) - \mathrm{i}F_1(\kappa_x, \kappa_y)}{p^2 - m^2 + \mathrm{i}\epsilon} \left[2k \cdot p \frac{\partial F_1(\kappa_x, \kappa_y)}{\partial \kappa_x} - p_\mu p_\nu h^{\mu\nu}(\kappa_x)\right]$$
$$= \int \frac{\mathrm{d}^4 p}{(2\pi)^4} \exp(-\mathrm{i}p \cdot (x - y))(1 - \exp(-\mathrm{i}F_1(\kappa_x, \kappa_y))). \tag{3.3}$$

The term in square brackets on the left-hand side of (3.3) is just equation (2.16). Therefore, the left-hand side of (3.3) vanishes if we take

$$F_1(\kappa_x, \kappa_y) = \frac{1}{2k \cdot p} \int_{\kappa_y}^{\kappa_x} \mathrm{d}\kappa\, p_\mu p_\nu h^{\mu\nu}(\kappa). \tag{3.4}$$

We now verify that the integral on the right-hand side of (3.3) also vanishes. We first separate the component of p_μ in the direction of k_μ [18], $p^\mu = p'^\mu + \alpha k^\mu$ where p'^μ spans a three-dimensional surface. Then the function $F_1(\kappa_x, \kappa_y)$ is independent of α and we can integrate over α, getting

$$\int \frac{\mathrm{d}\alpha}{2\pi} \exp(-\mathrm{i}\alpha(\kappa_x - \kappa_y))(1 - \exp(-\mathrm{i}F_1(\kappa_x, \kappa_y))) = \delta(\kappa_x - \kappa_y)(1 - \exp(-\mathrm{i}F_1(\kappa_x, \kappa_y))). \tag{3.5}$$

The right-hand side of (3.5) is obviously vanishing in view of (3.4). The Green's function has therefore the form

$$\Delta_F(x, y) = \int \frac{\mathrm{d}^4 p}{(2\pi)^4} \frac{\exp(-\mathrm{i}p \cdot (x-y))}{p^2 - m^2 + \mathrm{i}\epsilon} \exp\left(-\frac{\mathrm{i}}{2k \cdot p} \int_{\kappa_y}^{\kappa_x} \mathrm{d}\kappa\, h^{\mu\nu}(\kappa) p_\mu p_\nu\right). \tag{3.6}$$

This Feynman kernel was derived in paper [11] by looking at the problem in a semi-classical way and by evaluating the quantum mechanical path integral in a closed form.

We would now like to comment on the classical nature of the results so far obtained for the wavefunction (2.17) and the Green's function (3.6). Indeed, the classical action for a relativistic scalar particle in an external gravitational field

$$S = -m \int_{\tau_i}^{\tau_f} (g_{\mu\nu}(x)\dot{x}^\mu \dot{x}^\nu)^{1/2}\, \mathrm{d}\tau \tag{3.7}$$

is obtained by integrating a singular Lagrangian, giving rise to the first-class constraint $\chi = g^{\mu\nu}(x) p_\mu p_\nu - m^2$ and to a vanishing canonical Hamiltonian. An extended Hamiltonian proportional to the constraint, $\mathcal{H}_E(x, p) = \alpha_1 \chi$ is then defined. For a linearized gravitational wave field as (2.12) and in the harmonic gauge (2.11) we have

$$\mathcal{H}_E(x, p) = \alpha_1(\eta^{\mu\nu} p_\mu p_\nu - h^{\mu\nu}(x) p_\mu p_\nu - m^2). \tag{3.8}$$

Writing the corresponding covariant Hamilton–Jacobi equation

$$\eta^{\mu\nu} \frac{\partial S(x,p)}{\partial x^\mu} \frac{\partial S(x,p)}{\partial x^\nu} - h^{\mu\nu}(x) \frac{\partial S(x,p)}{\partial x^\mu} \frac{\partial S(x,p)}{\partial x^\nu} - m^2 = 0 \tag{3.9}$$

we shall look for a solution of the characteristic Hamilton function $S(x, p)$, expressed as a function of the coordinates x^μ and the initial momentum p^μ, of the general form

$$S(x, p) = -p \cdot x - S_1(\kappa_x). \tag{3.10}$$

From (3.9) and (2.13), we get the differential equation

$$2k \cdot p S_1'(\kappa_x) - h^{\mu\nu}(\kappa_x) p_\mu p_\nu = 0, \tag{3.11}$$

identical to equation (2.16) for $F(\kappa_x)$. Hence,

$$S(x, p) = -p \cdot x - \frac{1}{2k \cdot p} \int_{-\infty}^{k \cdot x} \mathrm{d}\kappa\, h^{\mu\nu}(\kappa) p_\mu p_\nu \tag{3.12}$$

so that the Volkov-like solution (2.17) turns out to be

$$\phi_p(x) = (2p_0 V)^{-\frac{1}{2}}\, \mathrm{e}^{\mathrm{i}S(x,p)}. \tag{3.13}$$

The WKB method applied to (2.14) therefore gives exact result in the first-order approximation, i.e., we have found an exact classical solution of equation (2.14). This circumstance had already been observed for scalar particles in [12]: we shall prove in the next section that the use of analytical mechanics involving skew-commuting variables allows us to generalize the mentioned property to spinning particles as well.

4. Dirac particle in an external gravitational wave: linear theory

We now study the same problem for a spin $\frac{1}{2}$ particle, considering the gravitational wave as an external field defined in a flat Minkowski background. (For the connection with the canonical intrinsic approach using Dirac equation in curved space–time, see [16].) Consequently, the minimal linearized Lagrangian describing the interaction of fermionic matter with a given gravitational field treated as a spin two external field in a Minkowski background is given by the expression [15, 16]

$$\mathcal{L} = \mathcal{L}_0 - \tfrac{1}{2} h_{\mu\nu}(x) T_0^{\mu\nu}(x), \tag{4.1}$$

in complete analogy with the scalar case described by equation (2.6). The Lagrangian density of the free fermionic matter $\mathcal{L}_0$ and the corresponding symmetrized energy–momentum tensor $T_0^{\mu\nu}(x)$ are given by the equations

$$\mathcal{L}_0 = \tfrac{1}{2}\bar\psi(x)(\mathrm{i}\gamma^\mu \overrightarrow{\partial_\mu} - m)\psi(x) + \tfrac{1}{2}\bar\psi(x)(-\mathrm{i}\gamma^\mu \overleftarrow{\partial_\mu} - m)\psi(x) \tag{4.2}$$

$$T_0^{\mu\nu}(x) = -\eta^{\mu\nu}\mathcal{L}_0 + \frac{\mathrm{i}}{4}(\bar\psi(x)(\gamma^\mu \overrightarrow{\partial^\nu} - \gamma^\mu \overleftarrow{\partial^\nu})\psi(x) + \bar\psi(x)(\gamma^\nu \overrightarrow{\partial^\mu} - \gamma^\nu \overleftarrow{\partial^\mu})\psi(x)). \tag{4.3}$$

The Lagrangian density $\mathcal{L}$ then becomes

$$\mathcal{L} = \left(1 + \frac{1}{2}h(x)\right)\mathcal{L}_0 - \frac{\mathrm{i}}{4}h^{\mu\nu}(x)\bar\psi(x)(\gamma_\mu \overrightarrow{\partial_\nu} - \gamma_\mu \overleftarrow{\partial_\nu})\psi(x) \tag{4.4}$$

and it can also be considered as obtained from the Lagrangian density established in general relativity for the interaction of the Dirac field with a prescribed gravitational field in the linear approximation [15, 16]. The Dirac equation we derive from (4.4) turns out to be [16]

$$\left(\left(1 + \frac{1}{2}h(x)\right)(\mathrm{i}\gamma^\mu\partial_\mu - m) - \frac{\mathrm{i}}{2}h^{\mu\nu}(x)\gamma_\mu\partial_\nu - \frac{\mathrm{i}}{4}\partial_\nu h^{\mu\nu}(x)\gamma_\mu + \frac{\mathrm{i}}{4}\partial_\mu h(x)\gamma^\mu\right)\psi(x) = 0 \tag{4.5}$$

or, multiplying it again by $\left(1 - \frac{1}{2}h(x)\right)$ and taking the linear terms in $h_{\mu\nu}(x)$,

$$\left((\mathrm{i}\gamma^\mu\partial_\mu - m) - \frac{\mathrm{i}}{2}h^{\mu\nu}(x)\gamma^\mu\partial_\nu - \frac{\mathrm{i}}{4}\partial_\nu h^{\mu\nu}(x)\gamma_\mu + \frac{\mathrm{i}}{4}\partial_\mu h(x)\gamma^\mu\right)\psi(x) = 0. \tag{4.6}$$

The harmonic condition (2.11) further simplifies the wave equation (4.6) giving the final form

$$\left(\mathrm{i}\gamma^\mu\partial_\mu - \frac{\mathrm{i}}{2}h^{\mu\nu}(x)\gamma_\mu\partial_\nu - m\right)\psi(x) = 0. \tag{4.7}$$

Moreover, since the external field represents a linearized gravitational wave, we assume again for $h_{\mu\nu}(x)$ the expression $h_{\mu\nu}(\kappa_x)$ given in (2.12) and (2.13).

Instead of solving the first-order Dirac equation (4.7), in analogy to what is done for the electromagnetic interactions [10], we find it easier to consider the second-order equation obtained by applying to (4.7) the operator $\left(\mathrm{i}\gamma^\mu\partial_\mu - \frac{\mathrm{i}}{2}h^{\mu\nu}(\kappa_x)\gamma_\mu\partial_\nu + m\right)$. By using the supplementary gauge condition (2.11) we finally get

$$\left(\partial^\mu\partial_\mu - h^{\mu\nu}(\kappa_x)\partial_\mu\partial_\nu + \frac{\mathrm{i}}{2}\sigma_{\mu\nu}(\partial^\mu h^{\nu\rho}(\kappa_x))\partial_\rho + m^2\right)\psi(x) = 0. \tag{4.8}$$

If we compare (4.8) with the scalar analogous equation (2.14), we can see that they are identical but for a term representing the spin–gravitational field interaction, as should be expected. Hence, we shall assume for the solutions of (4.8) a general form

$$\psi_{p,s}(x) = (2p_0V)^{-\frac{1}{2}}\,\mathrm{e}^{-\mathrm{i}p\cdot x}G(\kappa_x)u(p,s) \tag{4.9}$$

with the initial condition

$$\psi_{p,s}(x) \underset{x^0\to-\infty}{\longrightarrow} (2p_0V)^{-\frac{1}{2}}\,\mathrm{e}^{-\mathrm{i}p\cdot x}u(p,s) \tag{4.10}$$

where $u(p,s)$ is a constant spinor which is a solution of the corresponding free Dirac equation $(\gamma^\mu p_\mu - m)u(p,s) = 0$. Using the relations $k^2 = k_\mu h^{\mu\nu}(\kappa_x) = h^\mu_\mu(\kappa_x) = 0$ and $p^2 = m^2$, from (4.9) and (4.8) we get the first-order differential equation

$$2k\cdot pG'(\kappa_x) + \mathrm{i}\left(h^{\mu\nu}(\kappa_x)p_\mu p_\nu - \frac{1}{2}k^\mu p_\nu \frac{\mathrm{d}h^{\nu\rho}(\kappa_x)}{\mathrm{d}\kappa_x}\sigma_{\rho\mu}\right)G(\kappa_x) = 0 \tag{4.11}$$

with solution

$$G(\kappa_x) = \exp\left(-\frac{\mathrm{i}}{2k\cdot p}\int_{-\infty}^{\kappa_x} \mathrm{d}\kappa \left(h^{\mu\nu}(\kappa)p_\mu p_\nu - \frac{1}{2}k^\mu p_\nu \frac{\mathrm{d}h^{\nu\rho}(\kappa)}{\mathrm{d}\kappa}\sigma_{\rho\mu}\right)\right). \tag{4.12}$$

Taking into account that $(k^\mu p_\nu a^{\nu\rho}\sigma_{\rho\mu})^n = 0$ for $n > 1$, after some straightforward simplifications the Volkov-like solution (4.9) can finally be put into the form

$$\psi(x) = (2p_0V)^{-\frac{1}{2}}\left(1 + \frac{\mathrm{i}}{4}k^\mu p_\nu h^{\nu\rho}(\kappa_x)\sigma_{\rho\mu}\right) \times \exp\left(-\mathrm{i}p\cdot x - \frac{\mathrm{i}}{2k\cdot p}\int_{-\infty}^{\kappa_x}\mathrm{d}\kappa\, h^{\mu\nu}(\kappa)p_\mu p_\nu\right)u(p,s) \tag{4.13}$$

where, as usual, $\sigma_{\mu\nu} = \frac{\mathrm{i}}{2}[\gamma_\mu, \gamma_\nu]$ [19].

5. Green's function for the Dirac particle

We present here a direct calculation of the electron Green's function in the presence of an external gravitational plane wave. The Green's function satisfies the equation

$$\left(\mathrm{i}\gamma^\mu\partial_\mu - \frac{\mathrm{i}}{2}h^{\mu\nu}(\kappa_x)\gamma_\mu\partial_\nu - m\right)S_F(x,y) = \delta^4(x-y) \tag{5.1}$$

with the usual associated boundary conditions. As in section 4, we prefer working with a second-order equation. In fact, if we let

$$S_F(x,y) = \left(\mathrm{i}\gamma^\mu\partial_\mu - \frac{\mathrm{i}}{2}h^{\mu\nu}(\kappa_x)\gamma_\mu\partial_\nu + m\right)\Delta_F(x,y) \tag{5.2}$$

then, using again the gauge condition (2.11), we find that $\Delta_F(x,y)$ satisfies

$$\left(\partial^\mu\partial_\mu - h^{\mu\nu}(\kappa_x)\partial_\mu\partial_\nu + \frac{\mathrm{i}}{2}\sigma_{\mu\nu}(\partial^\mu h^{\nu\rho}(\kappa_x))\partial_\rho + m^2\right)\Delta_F(x,y) = -\delta^4(x-y). \tag{5.3}$$

Equation (5.3) is identical to the analogous equation (3.1) for the scalar particles but for the spin term $(\mathrm{i}/2)\sigma_{\mu\nu}(\partial^\mu h^{\nu\rho}(\kappa_x))\partial_\rho$. It is therefore natural to look for a solution of the form

$$\Delta_F(x,y) = \int \frac{\mathrm{d}^4p}{(2\pi)^4}\frac{\exp(-\mathrm{i}p\cdot(x-y))}{p^2 - m^2 + \mathrm{i}\epsilon}\exp(-\mathrm{i}F_1(\kappa_x,\kappa_y)N(\kappa_x,\kappa_y)) \tag{5.4}$$

where $F_1(\kappa_x,\kappa_y)$ is the same as in the scalar case (see equation (3.4)) and $N(\kappa_x,\kappa_y)$ is an unknown function to be determined. From (5.4) and (5.3), the equation for $N(\kappa_x,\kappa_y)$ becomes

$$\begin{aligned}\int \frac{\mathrm{d}^4p}{(2\pi)^4}&\frac{\exp(-\mathrm{i}p\cdot(x-y) - \mathrm{i}F_1(\kappa_x,\kappa_y))}{p^2 - m^2 + \mathrm{i}\epsilon}\\ &\times\left(\frac{1}{2}k^\mu p_\nu\frac{\mathrm{d}h^{\nu\rho}(\kappa_x)}{\mathrm{d}\kappa_x}\sigma_{\rho\mu}N(\kappa_x,\kappa_y) + 2\mathrm{i}k\cdot p\frac{\partial N(\kappa_x,\kappa_y)}{\partial\kappa_x}\right)\\ &= \int\frac{\mathrm{d}^4p}{(2\pi)^4}\,\mathrm{e}^{-\mathrm{i}p\cdot(x-y)}(1 - N(\kappa_x,\kappa_y)\exp(-\mathrm{i}F_1(\kappa_x,\kappa_y))).\end{aligned} \tag{5.5}$$

The left-hand side of equation (5.5) vanishes by choosing

$$N(\kappa_x,\kappa_y) = \exp\left(\frac{\mathrm{i}}{4k\cdot p}\int_{\kappa_y}^{\kappa_x}\mathrm{d}\kappa\, k^\mu p_\nu\frac{\mathrm{d}h^{\nu\rho}(\kappa)}{\mathrm{d}\kappa}\sigma_{\rho\mu}\right) \tag{5.6}$$

and, as in the scalar case, we easily verify that the right-hand side also vanishes. Thus,

$$\Delta_F(x,y) = \int \frac{\mathrm{d}^4 p}{(2\pi)^4}\left(1+\frac{\mathrm{i}}{4k\cdot p}k^\mu p_\nu h^{\nu\rho}(\kappa_x)\sigma_{\rho\mu}\right)$$
$$\times \exp\left(-\mathrm{i}p\cdot x - \frac{\mathrm{i}}{2k\cdot p}\int_{-\infty}^{k\cdot x}\mathrm{d}\kappa\, h^{\mu\nu}(\kappa)p_\mu p_\nu\right)$$
$$\frac{1}{p^2-m^2+\mathrm{i}\epsilon}\left(1-\frac{\mathrm{i}}{4k\cdot p}k^\mu p_\nu h^{\nu\rho}(\kappa_y)\sigma_{\rho\mu}\right)\exp\left(\mathrm{i}p\cdot y + \frac{\mathrm{i}}{2k\cdot p}\int_{-\infty}^{k\cdot y}\mathrm{d}\kappa\, h^{\mu\nu}(\kappa)p_\mu p_\nu\right) \tag{5.7}$$

is a solution of the inhomogeneous second-order Dirac equation (5.3) and from equation (5.2) we finally get for the Feynman propagator $S_F(x,y)$ the expression

$$S_F(x,y) = \left(\mathrm{i}\gamma^\mu\partial_\mu - \frac{\mathrm{i}}{2}h^{\mu\nu}(\kappa_x)\gamma_\mu\partial_\nu + m\right)\Delta_F(x,y)$$
$$= \int \frac{\mathrm{d}^4 p}{(2\pi)^4}\left(1+\frac{\mathrm{i}}{4k\cdot p}k^\mu p_\nu h^{\nu\rho}(\kappa_x)\sigma_{\rho\mu}\right)$$
$$\times \frac{\gamma_\mu p^\mu + m}{p^2-m^2+\mathrm{i}\epsilon}\left(1-\frac{\mathrm{i}}{4k\cdot p}k^\mu p_\nu h^{\nu\rho}(\kappa_y)\sigma_{\rho\mu}\right)$$
$$\times \exp\left(-\mathrm{i}p\cdot(x-y) - \frac{\mathrm{i}}{2k\cdot p}\int_{\kappa_y}^{\kappa_x}\mathrm{d}\kappa\, h^{\mu\nu}(\kappa)p_\mu p_\nu\right). \tag{5.8}$$

As we have already discussed for the scalar particle in section 3, it would be interesting to show the *pseudoclassical* nature of the results obtained for the Dirac particle too. We start with the pseudoclassical Lagrangian for a spin $\frac{1}{2}$ particle in an arbitrary external gravitational field [9]

$$\mathcal{L} = -\frac{\mathrm{i}}{2}g_{\mu\nu}(x)\zeta^\mu\left(\dot\zeta^\nu + \Gamma^\nu_{\lambda\rho}(x)\dot x^\rho\zeta^\lambda\right) - \frac{\mathrm{i}}{2}\zeta_5\dot\zeta_5 - m\left(g_{\mu\nu}(x)(\dot x^\mu - \frac{\mathrm{i}}{m}\zeta^\mu\dot\zeta_5)(\dot x^\nu - \frac{\mathrm{i}}{m}\zeta^\nu\dot\zeta_5)\right)^{\frac{1}{2}} \tag{5.9}$$

where $\Gamma^\nu_{\lambda\rho}(x)$ are the usual Christoffel symbols and the Grassmann variables ζ^μ are quite naturally introduced, since in quantization they correspond to the use of the spin matrices conform to metric γ^μ (see for instance [20] and references therein). By introducing the *vierbein* field $G^A_\mu(x)$ and its inverse $H^\mu_B(x)$

$$g_{\mu\nu}(x) = \eta_{AB}G^A_\mu(x)G^B_\nu(x), \qquad g^{\mu\nu}(x) = \eta^{AB}(x)H^\mu_A(x)H^\nu_B(x) \tag{5.10}$$

$$H^\mu_A(x)G^A_\nu(x) = \delta^\mu_\nu, \qquad H^\mu_A(x)G^B_\mu(x) = \delta^B_A \tag{5.11}$$

we can transform a world 4-vector ζ^μ into a 'local' one and we can also introduce Grassmann variables ξ^A corresponding, after quantization, to the usual Dirac matrices (flat space spin matrices)

$$\xi^A = G^A_\mu(x)\zeta^\mu, \qquad \zeta^\mu = H^\mu_A(x)\xi^A, \qquad \xi_5 = \zeta_5. \tag{5.12}$$

The singular Lagrangian (5.9) gives rise to the two first-class constraints $\chi_D = \mathcal{P}_\mu\zeta^\mu - \mathrm{i}m(\pi_5 - (\mathrm{i}/2)\zeta_5)$ and $\chi = g^{\mu\nu}(x)\mathcal{P}_\mu\mathcal{P}_\nu - m^2$, where P_μ is the canonical momentum and $\mathcal{P}_\mu = P_\mu - (\mathrm{i}/2)g_{\rho\nu}(x)\zeta^\rho\Gamma^\nu_{\mu\lambda}(x)\zeta^\lambda$ is the mechanical one. We also have a second-class constraint

$$\pi_\mu = \frac{\mathrm{i}}{2}g_{\mu\nu}(x)\zeta^\nu \tag{5.13}$$

and, in addition, we can further require [3, 9]

$$\pi_5 = -\frac{\mathrm{i}}{2}\zeta_5. \tag{5.14}$$

Relations (5.13), (5.14) form a set of second-class constraints whose relevant Dirac brackets are

$$\{\zeta_5, \zeta_5\} = -\mathrm{i}, \qquad \{\zeta^\mu, \zeta^\nu\} = \mathrm{i}g^{\mu\nu}(x), \qquad \{\xi^A, \xi^B\} = \mathrm{i}\eta^{AB}. \tag{5.15}$$

Some remarks on the quantization are now in order. The Dirac brackets (5.15) give the anticommutation rules

$$[\widehat{\xi}_5, \widehat{\xi}_5]_+ = 1, \qquad [\widehat{\xi}^A, \widehat{\xi}^B]_+ = -\eta^{AB}, \qquad [\widehat{\zeta}^\mu, \widehat{\zeta}^\nu]_+ = -g^{\mu\nu}(x). \tag{5.16}$$

The first and the second relation can be satisfied by putting

$$\widehat{\xi}_5 = 2^{-\frac{1}{2}}\gamma_5, \qquad \widehat{\xi}^A = 2^{-\frac{1}{2}}\gamma_5\gamma^A \tag{5.17}$$

where γ_5 and γ^A are the usual flat space Dirac matrices. We must now choose a representation for the algebra of the operators $\widehat{\zeta}^\mu$. A possible choice which will correspond to the use of the spin matrices conform to metric is

$$\widehat{\zeta}^\mu = H^\mu_A(x)\widehat{\xi}^A = 2^{-\frac{1}{2}}\gamma_5\gamma^\mu \tag{5.18}$$

where

$$[\gamma^\mu, \gamma^\nu]_+ = 2g^{\mu\nu}(x). \tag{5.19}$$

As usual, the extended Hamiltonian $\mathcal{H}_E$ is written as a linear combination of the first-class constraints with arbitrary coefficients and the simplest choice appears to be

$$\mathcal{H}_E = \alpha_1(g^{\mu\nu}(x)\mathcal{P}_\mu\mathcal{P}_\nu - m^2). \tag{5.20}$$

The constraint χ_D becomes $\chi_D = \mathcal{P}_\mu\zeta^\mu - m\zeta_5$, once equation (5.14) is accounted for. It could be used, after quantization, for constructing the spinorial states. Considering again the restriction to a linearized gravitational wave field in the harmonic gauge, (2.12), (2.13) and taking into account the linearized form $H^\mu_A(\kappa_x) = \delta^\mu_A - (1/2)h^\mu_A(\kappa_x)$, $G^A_\mu(\kappa_x) = \delta^A_\mu + (1/2)h^A_\mu(\kappa_x)$ of the vierbein fields, we finally get

$$\mathcal{H}_E = \alpha_1\left(\eta^{\mu\nu}P_\mu P_\nu - h^{\mu\nu}(\kappa_x)P_\mu P_\nu + \mathrm{i}P_\rho\frac{\partial h^{\rho\nu}(\kappa_x)}{\partial x_\mu}\xi_\mu\xi_\nu - m^2\right). \tag{5.21}$$

Therefore the variables ζ^μ satisfy the linearized relations

$$\zeta^\mu = \xi^\mu - \tfrac{1}{2}h^\mu_A(x)\xi^A \tag{5.22}$$

and we finally obtain the following equation for the Hamilton's characteristic function $S(x)$:

$$\eta^{\mu\nu}\frac{\partial S(x)}{\partial x^\mu}\frac{\partial S(x)}{\partial x^\nu} - h^{\mu\nu}(x)\frac{\partial S(x)}{\partial x^\mu}\frac{\partial S(x)}{\partial x^\nu} - \mathrm{i}\frac{\partial S(x)}{\partial x^\rho}\frac{\partial h^{\rho\nu}(x)}{\partial x_\mu}\xi_\mu\xi_\nu - m^2 = 0. \tag{5.23}$$

The solution (3.12) obtained in the scalar case suggests, for the pseudoclassical Hamilton characteristic function, the ansatz

$$S(x) = -p\cdot x - \frac{1}{2k\cdot p}\int_{-\infty}^{\kappa_x}\mathrm{d}\kappa\; p_\mu p_\nu h^{\mu\nu}(\kappa) + S_D(\kappa_x). \tag{5.24}$$

The unknown function S_D will satisfy the first-order differential equation

$$2k\cdot p\frac{\mathrm{d}S_D(\kappa_x)}{\mathrm{d}\kappa_x} + \mathrm{i}k^\mu p_\nu\frac{\mathrm{d}h^{\nu\rho}(\kappa_x)}{\mathrm{d}\varphi_x}\xi_\rho\xi_\mu = 0 \tag{5.25}$$

with solution

$$S_D(x) = -\frac{\mathrm{i}}{2k \cdot p} \int_{-\infty}^{\kappa_x} \mathrm{d}\kappa\, k^\mu p_\nu \frac{\mathrm{d}h^{\nu\rho}(\kappa)}{\mathrm{d}\kappa} \xi_\rho \xi_\mu. \tag{5.26}$$

It can easily be seen that the operator $\widehat{S}$ obtained upon quantization—through the use of equations (5.17)—of the pseudoclassical $S(x)$ given in (5.24) can be represented as

$$\widehat{S} = -p \cdot x - \frac{1}{2k \cdot p} \int_{-\infty}^{\kappa_x} \mathrm{d}\kappa \left(h^{\mu\nu}(\kappa) p_\mu p_\nu - \frac{1}{2} k^\mu p_\nu \frac{\mathrm{d}h^{\nu\rho}(\kappa)}{\mathrm{d}\kappa} \sigma_{\rho\mu} \right). \tag{5.27}$$

Therefore by applying the operator $(2p_0 V)^{-\frac{1}{2}}\, \mathrm{e}^{\mathrm{i}\widehat{S}}$ to a free constant spinor $u(p,s)$ we see that the Volkov-like solution (4.13) will be written as follows

$$\psi(x) = (2p_0 V)^{-\frac{1}{2}}\, \mathrm{e}^{\mathrm{i}\widehat{S}} u(p,s) \tag{5.28}$$

completely showing the semi-classical nature of the solution.

References

[1] Golfand Yu A and Lichtman E P 1971 *JETP Lett.* **13** 452
Volkov D V and Akulov V P 1973 *Phys. Lett.* B **46** 109
Wess J and Zumino B 1974 *Nucl. Phys.* B **70** 39
Salam A and Strathdee J 1974 *Nucl. Phys.* B **76** 477
[2] Casalbuoni R 1976 *Nuovo Cimento* A **33** 115
Casalbuoni R 1976 *Nuovo Cimento* A **33** 389
[3] Barducci A, Casalbuoni R and Lusanna L 1976 *Nuovo Cimento* A **35** 377
[4] Brink L, Deser S, Zumino B, Di Vecchia P and Howe P 1976 *Phys. Lett.* B **46** 435
Berezin F A and Marinov M S 1977 *Ann. Phys., NY* **104** 336
[5] Giachetti R, Ragionieri R and Ricci R 1981 *J. Diff. Geom.* **16** 297
Giachetti R and Ricci R 1986 *Adv. Math.* **62** 84
[6] Barducci A, Casalbuoni R and Lusanna L 1977 *Nucl. Phys.* B **124** 93
[7] Bordi F and Casalbuoni R 1980 *Phys. Lett.* B **93** 308
Barducci A, Bordi F and Casalbuoni R 1981 *Nuovo Cimento* B **64** 287
[8] Barducci A and Giachetti R 2003 *J. Phys. A: Math. Gen.* **36** 8129
[9] Barducci A, Casalbuoni R and Lusanna L 1977 *Nucl. Phys.* B **124** 521
[10] Volkov D M 1935 *Z. Phys.* **94** 25
Berestetski V, Lifchitz E and Pitayevski L 1973 *Théorie Quantique Rélativiste* (Moscow: Mir)
[11] Barducci A and Giachetti R 1976 *Lett. Nuovo Cimento* **15** 309
[12] Garriga J and Verdaguer E 1991 *Phys. Rev.* D **43** 391
[13] Bini D and Ferrari V 1995 *Int. J. Mod. Phys.* D **4** 291
[14] Kessari S, Singh D, Tucker R W and Wang C 2002 *Class. Quantum Grav.* **19** 4943
[15] Weinberg S 1972 *Gravitation and Cosmology* (New York: Wiley)
[16] Bordé C J, Houard J C and Karasiewicz A 2001 *Lecture Notes in Physics* vol 562 p 403
[17] Schwinger J 1951 *Phys. Rev.* **82** 664
[18] Brown L S and Kibble T W B 1964 *Phys. Rev.* A **133** 705
[19] Bjorken J D and Drell S D 1964 *Relativistic Quantum Mechanics* (New York: McGraw-Hill)
[20] Brill D R and Wheeler J A 1957 *Rev. Mod. Phys.* **29** 465